

The Embedding of Superstring Backgrounds in Einstein Gravity

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Abstract

A theorem of differential geometry is employed to locally embed a wide class of superstring backgrounds that admit a covariantly constant null Killing vector field in eleven-dimensional, Ricci-flat spaces. Included in this class are exact type IIB superstring backgrounds with non-trivial Ramond-Ramond fields and a class of supersymmetric string waves. The embedding spaces represent exact solutions to eleven-dimensional, vacuum Einstein gravity. A solution of eleven-dimensional supergravity is also embedded in a twelve-dimensional, Ricci-flat space.

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Embedding theorems of differential geometry are important from both a mathematical and physical point of view. They relate higher- and lower-dimensional theories of gravity and give rise to classification schemes for different spacetimes [1]. They can also lead to new solutions of Einstein's field equations [2]. A well known theorem states that any analytic, n -dimensional Riemannian manifold can be locally and isometrically embedded in a pseudo-Euclidean space of dimension $n \leq N \leq n(n+1)/2$ [3]. It is also known that the pseudo-Euclidean space must have dimension $N \geq n+2$ if the embedded space is Ricci-flat [4].

On the other hand, there is a theorem due to Campbell that states that any analytic, n -dimensional Riemannian manifold can be locally embedded in a *Ricci-flat*, $(n+1)$ -dimensional, Riemannian space, where the extra dimension may be either space-like or time-like [5, 6, 7]. Since the embedding space is Ricci-flat, it represents an exact solution to the vacuum, Einstein field equations in $(n+1)$ dimensions. One interesting application of this theorem, therefore, is to generate new solutions to vacuum general relativity by considering the embedding of lower-dimensional solutions.

The embedding of four-dimensional electromagnetic and gravitational plane waves in five dimensions was recently established by application of Campbell's theorem [8]. The purpose of the present paper is to employ the theorem to embed exact ten-dimensional superstring backgrounds in eleven-dimensional Einstein gravity. It is widely thought that superstring theory represents a consistent quantum theory of gravity and the study of classical solutions to the string equations of motion that are exact to all orders in the inverse string tension, α' , is therefore important.

We consider spacetimes that admit a covariantly constant null Killing vector field. The most general, n -dimensional spacetime admitting such a field is the Brinkmann metric [9]¹:

$$ds^2 = 2dudv + A_\mu(u, x^i)dx^\mu du - g_{ij}(u, x^i)dx^i dx^j, \quad (1)$$

where the Killing vector satisfies $l_{\mu;\nu} = 0$ and $l^\mu l_\mu = 0$ and the light-cone coordinates $\{u, v\}$ are defined by $l_\mu \equiv \partial_\mu u$ and $l^\mu \partial_\mu v = 1$, respectively. The vector function, A_μ ,

¹Greek indices vary from $\mu = (0, 1, \dots, n-1)$, lower case Latin indices run from $i = (2, 3, \dots, n-1)$ and upper case Latin indices take values in the range $A = (0, 1, \dots, n)$. The signature of spacetime is $(+, -, -, \dots)$, a semicolon denotes covariant differentiation and ∂_μ denotes partial differentiation with respect to x^μ .

satisfies $l^\mu A_\mu = 0$ and has arbitrary dependence on u and x^i . The function $g_{ij}(u, x^i)$ is symmetric and positive definite. When the functional forms of the metric components satisfy appropriate conditions, Eq. (1) describes a wide class of exact superstring backgrounds [10, 11, 12, 13, 14, 15, 16, 17]. This includes generalized string plane waves [11, 12] and the F^- and K^- models [13, 14]. (For a recent review of the different types of known exact solutions see, e.g., Ref. [16]).

We begin by briefly reviewing Campbell's theorem and proceed to establish the embedding of Brinkmann spacetimes. We then apply the embedding procedure directly to exact superstring backgrounds. In particular, an embedding for a general class of type IIB backgrounds with non-trivial Ramond-Ramond (RR) fields is found. Similar embeddings are also determined for supersymmetric string waves and solutions to eleven-dimensional supergravity.

In general an analytic, n -dimensional Riemannian manifold with metric $^{(n)}g_{\alpha\beta}(x^\mu)$ and line element $^{(n)}ds^2 = ^{(n)}g_{\alpha\beta}(x^\mu)dx^\alpha dx^\beta$ may be locally embedded in a $(n+1)$ -dimensional manifold with a line element

$$^{(n+1)}ds^2 = g_{\alpha\beta}(x^\mu, \psi)dx^\alpha dx^\beta + \epsilon\varphi^2(x^\mu, \psi)d\psi^2, \quad \epsilon = \pm 1, \quad (2)$$

if the metric coefficients $g_{\alpha\beta} = g_{\alpha\beta}(x^\mu, \psi)$ and $\varphi = \varphi(x^\mu, \psi)$ satisfy certain restrictions. The problem of embedding the n -dimensional manifold in $(n+1)$ dimensions is reduced to determining the appropriate functional forms for these components. This is achieved by introducing the set of functions $\Omega_{\alpha\beta} = \Omega_{\alpha\beta}(x^\mu, \psi)$ that satisfy the conditions

$$\Omega_{\alpha\beta} = \Omega_{\beta\alpha} \quad (3)$$

$$\Omega^\alpha_{\beta;\alpha} = \partial_\beta \Omega \quad (4)$$

$$\Omega_{\alpha\beta}\Omega^{\alpha\beta} - \Omega^2 = -\epsilon^{(n)}R \quad (5)$$

on an arbitrary hypersurface $\psi = \psi_0 = \text{constant}$, where

$$\Omega^\alpha_{\beta} \equiv ^{(n)}g^{\alpha\lambda}\Omega_{\lambda\beta}, \quad \Omega \equiv ^{(n)}g^{\alpha\beta}\Omega_{\alpha\beta}, \quad ^{(n)}R \equiv ^{(n)}R_{\alpha\beta}^{(n)}g^{\alpha\beta} \quad (6)$$

and $^{(n)}R_{\alpha\beta}$ is the Ricci curvature tensor of the manifold calculated from $^{(n)}g_{\alpha\beta}$. We also require that $g_{\alpha\beta}$ and $\Omega_{\alpha\beta}$ satisfy the coupled, partial differential equations

$$\frac{\partial g_{\alpha\beta}}{\partial \psi} = -2\varphi\Omega_{\alpha\beta} \quad (7)$$

$$\frac{\partial \Omega^\alpha{}_\beta}{\partial \psi} = \varphi \left(-\epsilon^{(n)} R^\alpha{}_\beta + \Omega \Omega^\alpha{}_\beta \right) + \epsilon g^{\alpha\lambda} \varphi_{;\lambda\beta} \quad (8)$$

and, moreover, that the components $g_{\alpha\beta}$ reduce to the n -dimensional metric on the hypersurface $\psi = \psi_0$, i.e., that

$$g_{\alpha\beta}(x^\mu, \psi_0) = {}^{(n)}g_{\alpha\beta}(x^\mu). \quad (9)$$

It can then be shown that Eqs. (3)–(8) correspond to the $(n+1)$ -dimensional field equations of vacuum Einstein gravity,

$${}^{(n+1)}R_{AB}(x^\mu, \psi_0) = 0, \quad (10)$$

when Eqs. (7) and (8) are evaluated on the $\psi = \psi_0$ hypersurface [7, 18]. However, if Eqs. (7) and (8) are valid in general, it can also be shown that Eqs. (3)–(5) are satisfied for all ψ in the neighbourhood of ψ_0 [5]. Thus, the Ricci tensor of the $(n+1)$ -dimensional space vanishes for all ψ in the neighbourhood of ψ_0 , and since ψ_0 is arbitrary, this implies that the local embedding of the n -dimensional metric ${}^{(n)}g_{\alpha\beta}$ in a $(n+1)$ -dimensional, Ricci-flat space is given by Eq. (2). Thus, the embedding is established by solving the set of equations (3)–(9).

We assume that $n = 10$ and that the vector and tensor functions, $A_\mu(u, x^i)$ and $g_{ij}(u, x^i)$, in Eq. (1) are given by

$$A_u(u, x^i) \equiv \frac{1}{2}K(u, x^i) = \frac{1}{2}K_0(u, x^i) + \frac{1}{2}K_{ij}(u)x^i x^j$$

$$A_v \equiv 0, \quad A_i(u, x^j) \equiv -\frac{1}{2}M_{ij}(u)x^j \quad (11)$$

and

$$g_{ij}(u) \equiv f(u)\delta_{ij}, \quad (12)$$

respectively, where δ_{ij} denotes the eight-dimensional Kronecker delta and $M_{ij}(u) \equiv \partial_i A_j - \partial_j A_i$ represents an effective field strength of the couplings A_i . The quantities M_{ij} , K_{ij} and f are arbitrary functions of u only. The antisymmetric nature of M_{ij} implies that A_1 is independent of x^1 , etc. The function K_0 is an arbitrary solution to the eight-dimensional Laplace equation, $\Delta K_0 = 0$, where the Laplacian $\Delta \equiv \delta^{ij}\partial_i\partial_j$ is taken over the eight transverse directions. The only non-zero components of the Riemann tensor are ${}^{(10)}R_{uiuj}$

and it follows that the one non-trivial component of the Ricci tensor is

$${}^{(10)}R_{uu} = -4f^{-1}\partial_u^2 f + 2f^{-2}(\partial_u f)^2 + \frac{1}{4}f^{-1}\Delta K + \frac{1}{16}f^{-2}M^2, \quad (13)$$

where $M^2 \equiv M_{ij}M_{ab}\delta^{ia}\delta^{jb}$. Moreover, since ${}^{(10)}g^{uu} = 0$, the Ricci scalar curvature of the spacetime (1) vanishes, ${}^{(10)}R = 0$.

Given the ansatz (11) and (12), we now establish the local embedding of the general class of spacetimes (1) in eleven-dimensional, vacuum Einstein gravity. The vanishing of the Ricci curvature scalar of the ten-dimensional metric implies that $\Omega_{\alpha\beta} = 0$ is a consistent solution to Eqs. (3)–(5). Eq. (7) then implies that $\partial g_{\alpha\beta}/\partial\psi = 0$. Thus, the components $g_{\alpha\beta}$ of the eleven-dimensional embedding metric are independent of ψ and Eq. (9) is satisfied by choosing $g_{\alpha\beta}(x^\mu) = {}^{(10)}g_{\alpha\beta}(x^\mu)$. The embedding is therefore completed once Eq. (8) has been solved and this equation represents the set of differential equations

$${}^{(10)}g^{\alpha\lambda}\varphi_{;\lambda\beta} - {}^{(10)}R^\alpha{}_\beta\varphi = 0. \quad (14)$$

It is interesting that Eq. (14) is independent of ϵ , i.e., on whether the eleventh dimension is space-like or time-like.

All the components of ${}^{(10)}R^\alpha{}_\beta$ vanish apart from ${}^{(10)}R^v{}_u = {}^{(10)}R_{uu}$. The form of $K(u, x^i)$ in Eq. (11) is such that ${}^{(10)}R^v{}_u$ is a function of the light-cone coordinate u only. This implies that it is consistent to assume that $\varphi = \varphi(u)$. It then follows that all the components of Eq. (14) are trivially satisfied unless $(\alpha, \beta) = (v, u)$. In this later case, Eq. (14) simplifies to the second-order, ordinary differential equation

$$\left[\frac{d^2}{du^2} - {}^{(10)}R_{uu}(u) \right] \varphi(u) = 0. \quad (15)$$

Thus, the embedding of the class of ten-dimensional spacetimes (1), (11) and (12) is given by

$${}^{(11)}ds^2 = 2dudv - \frac{1}{2}M_{ij}(u)x^j dxdx^i + \frac{1}{2}K(u, x^i)du^2 - f(u)\delta_{ij}dx^i dx^j + \epsilon\varphi^2(u)d\psi^2, \quad (16)$$

where $\varphi = \varphi(u)$ solves Eq. (15). It may be verified by direct calculation of the eleven-dimensional Ricci curvature tensor that the manifold with metric (16) is indeed Ricci-flat when Eq. (15) is satisfied. We emphasize that the embedding is valid for *arbitrary* functions

$\{K_{ij}(u), M_{ij}(u), f(u)\}$. The function $K_0(u, x^i)$ is also arbitrary, subject to the condition $\Delta K_0 = 0$.

We now apply these results to the type II superstring theories. The zero-slope limit of the type IIA (IIB) superstring is $N = 2$, $n = 10$ non-chiral (chiral) supergravity [19, 20]. The action for the Neveu-Schwarz/Neveu-Schwarz (NS-NS) sector of these theories in the string frame is [21]

$$S = \int d^{10}x \sqrt{|^{(10)}g|} e^{-\Phi} \left[^{(10)}R + (\nabla\Phi)^2 - \frac{1}{3} \left(H^{(1)} \right)^2 \right], \quad (17)$$

where the graviton, $^{(10)}g_{\mu\nu}$, dilaton, Φ , and antisymmetric two-form potential, $B_{\mu\nu}^{(1)}$, are the massless excitations. The field strength of the two-form is given by $H_{\mu\nu\lambda}^{(1)} \equiv \partial_{[\mu} B_{\nu\lambda]}^{(1)}$ and $^{(10)}g \equiv \det(^{(10)}g_{\mu\nu})$. The background field equations derived from action (17) correspond to the conditions for the one-loop β -functions of the massless excitations to vanish. These may be solved by the *ansatz* [15, 16]

$$B_{\mu\nu}^{(1)} = 3B_{[\mu}^{(1)} l_{\nu]}, \quad B_v^{(1)} = 0, \quad B_i^{(1)}(u, x^j) = -\frac{1}{2} N_{ij}(u) x^j, \quad \Phi = \Phi(u) \quad (18)$$

when the metric is given by Eqs. (1), (11) and (12) with $f(u) = 1$. The field strength $N_{ij}(u) \equiv \partial_i B_j - \partial_j B_i$ is a function of u only. The β_{uu} -component of the graviton β -function then takes the form

$$\Delta K + \frac{1}{4} M^2 - N^2 + 4\partial_u^2 \Phi = 0, \quad (19)$$

where $N^2 \equiv N_{ij} N_{ab} \delta^{ai} \delta^{bj}$ and this may be solved by choosing an appropriate functional form for $\text{Tr} K_{ij}(u)$. All other components of the β -functions vanish identically. We refer to the class of superstring backgrounds that satisfy Eqs. (1), (11), (12), (18) and (19) as the ‘NS-NS backgrounds’. A geometrical argument may be employed to show that these backgrounds are exact to all orders in the inverse string tension [12, 17]. The existence of the covariantly constant null Killing vector field implies that the Riemann curvature tensor is orthogonal to l^μ . The same property is exhibited by $H_{\mu\nu\lambda}^{(1)}$ and the derivatives of the dilaton field. This implies that all higher-order terms in the equations of motion vanish identically.

We conclude immediately, therefore, that the embedding of these exact NS-NS backgrounds in eleven-dimensional, Ricci-flat manifolds is given by Eqs. (15) and (16) and, in

general, substitution of Eq. (19) into Eq. (13) implies that

$${}^{(10)}R_{uu}(u) = \frac{1}{4}N^2(u) - \partial_u^2\Phi. \quad (20)$$

As a specific example, we consider the case:

$$N(u) \equiv \text{constant}, \quad \Phi(u) = a_0 + a_1u + a_2u^2/2, \quad (21)$$

where $\{a_i\}$ are arbitrary constants. This implies that ${}^{(10)}R_{uu}$ is itself a constant and the general solution to Eq. (15) is therefore given by

$$\varphi(u) = A \exp \left[\sqrt{{}^{(10)}R_{uu}} u \right] + B \exp \left[-\sqrt{{}^{(10)}R_{uu}} u \right], \quad (22)$$

where A and B are arbitrary constants. We emphasize that this embedding is valid for arbitrary functions $M_{ij}(u)$. In principle, a large class of non-trivial embeddings could be analytically calculated in this fashion for different functional forms of ${}^{(10)}R_{uu}$.

The global $\text{SL}(2, R)$ symmetry [22, 23] of the ten-dimensional type IIB superstring has recently been employed to generate new, exact type IIB backgrounds with non-trivial RR fields from the above NS-NS backgrounds [17]. These backgrounds are interesting because it is difficult to perturbatively calculate higher-order terms in the type IIB theory due to the specific coupling of the RR sector [24]. This sector consists of a pseudo-scalar axion field, χ , a two-form potential, $B_{\mu\nu}^{(2)}$, and a four-form potential, $D_{\mu\nu\lambda\kappa}$ [19, 21]. The complex scalar field $\lambda \equiv \chi + ie^{-\Phi/2}$ parametrizes the coset $\text{SL}(2, R)/\text{U}(1)$ and transforms to $\tilde{\lambda} = (a\lambda + b)/(c\lambda + d)$ under a global $\text{SL}(2, R)$ transformation, where $ad - bc = 1$. The NS-NS and RR two-forms transform as a doublet, $\tilde{B}_{\mu\nu}^{(i)} = (\Theta^T)^{-1} B_{\mu\nu}^{(i)}$, where $B_{\mu\nu}^{(i)}$ represents a two-component vector and

$$\Theta \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (23)$$

is an $\text{SL}(2, R)$ matrix. The four-form transforms as a singlet, $\tilde{D}_{\mu\nu\lambda\kappa} = D_{\mu\nu\lambda\kappa}$, and the metric transforms to

$${}^{(10)}\tilde{g}_{\mu\nu} = \exp \left[(\tilde{\Phi} - \Phi)/4 \right] {}^{(10)}g_{\mu\nu}. \quad (24)$$

The NS-NS backgrounds represent solutions to the type IIB string equations of motion with $\chi = B_{\mu\nu}^{(2)} = D_{\mu\nu\lambda\kappa} = 0$. This implies that $\lambda = ie^{-\Phi/2}$ and applying a general $\text{SL}(2, R)$

transformation to these backgrounds generates a new type IIB background [17]:

$${}^{(10)}ds_{\text{IIB}}^2 = f(u) \left[2dudv + \frac{1}{2}K du^2 + A_i dx^i du - \delta_{ij} dx^i dx^j \right], \quad (25)$$

where

$$f(u) \equiv \left[d^2 + c^2 e^{-\Phi(u)} \right]^{1/2}. \quad (26)$$

This transformation generates non-trivial χ and $B_{\mu\nu}^{(2)}$ fields. Redefining the null variable $U \equiv \int^u du' f(u')$ and relabelling it as u then implies that the metric (25) takes the form

$${}^{(10)}ds_{\text{IIB}}^2 = 2dudv + \frac{1}{2}\hat{K}(u, x^i) du^2 + A_i(u, x^i) dx^i du - f(u) \delta_{ij} dx^i dx^j, \quad (27)$$

where

$$\hat{K} \equiv f^{-1}K. \quad (28)$$

It is important that \hat{K} is a quadratic function of the transverse coordinates x^i , because in this case it can be shown that the type IIB backgrounds (27) are exact to all orders in α' [17]. Furthermore, the metric (27) is precisely of the form given in Eq. (1), where the metric components satisfy Eqs. (11) and (12) and \hat{K} is identified with K . Thus, the only non-vanishing component of the Ricci-tensor of the spacetime (27) is ${}^{(10)}R_{uu} = {}^{(10)}R_u^v$ and substitution of Eqs. (19) and (28) into Eq. (13) implies that

$${}^{(10)}R_{uu}(u) = -4f^{-1}\partial_u^2 f + 2f^{-2}(\partial_u f)^2 + \frac{1}{4}f^{-2}(N^2 - 4\partial_u^2 \Phi). \quad (29)$$

Since this component is a function of u only, we may conclude that an embedding in an eleven-dimensional, Ricci-flat space of the general class of type IIB backgrounds (27) with excited RR fields is given by Eqs. (15) and (16).

It is interesting to consider the background generated by the specific $\text{SL}(2, R)$ transformation where $a = d = 0$ and $c = -b = 1$. Since the RR axion is initially trivial ($\chi = 0$), this transformation changes the sign of the dilaton field, $\tilde{\Phi} = -\Phi$, and inverts the string coupling, $\tilde{g}_s^2 \equiv e^{\tilde{\Phi}} = g_s^{-2} = e^{-\Phi}$. It therefore maps the strongly-coupled regime of the theory onto the weakly-coupled regime, and vice-versa. It also directly interchanges the NS-NS and RR two-forms and leaves $\chi = 0$. Since the RR two-form is initially zero, the new background has a trivial NS-NS two-form and non-trivial RR two-form.

Substituting Eq. (26) into Eq. (29) then implies that

$${}^{(10)}R_{uu} = -\frac{1}{2}(\partial_u \Phi)^2 + 2\partial_u^2 \Phi + \left(\frac{N^2}{4} - \partial_u^2 \Phi\right) e^\Phi \quad (30)$$

and Eq. (15) can now be solved exactly if Eq. (21) is satisfied and the dilaton is a linear function of u ($a_2 = 0$). If we further assume for simplicity that $a_0 = 0$, substitution of Eq. (30) into Eq. (15) implies that

$$\left[\frac{d^2}{dz^2} + \frac{1}{z} \frac{d}{dz} + \frac{a_1^2}{2} z^{-2} - \frac{N^2}{4} z^{a_1-2} \right] \varphi = 0, \quad (31)$$

where $z \equiv e^u$. Modulo a constant of proportionality, the general solution to Eq. (31) is given by

$$\varphi = Z_p \left(N e^{a_1 u/2} / a_1 \right), \quad (32)$$

where Z_p represents a linear combination of modified Bessel functions of the first and second kind of order $p = \pm\sqrt{2}i$. Identifying Z with a modified function of the second kind, for example, implies that φ oscillates in the asymptotic limit $u \rightarrow -\infty$ and becomes exponentially damped when $u \rightarrow +\infty$ [25]. This differs from the purely oscillatory or exponential behaviour of Eq. (22) for the NS–NS background.

The RR fields χ and $B_{\mu\nu}^{(2)}$ do not have an eleven-dimensional supergravity origin. However, Campbell's theorem implies that they may have a *geometrical* origin in the sense that all matter fields can in principle be geometrized in terms of higher dimensions. This is closely related to Wesson's interpretation of Kaluza–Klein gravity [26, 27]. In this picture, it can be shown that five-dimensional, vacuum Einstein gravity gives rise to four-dimensional gravity with a *general* energy–momentum tensor if one relaxes the condition that physical quantities be independent of the extra dimension, as was assumed in the original Kaluza–Klein theory. Thus, derivatives with respect to the fifth coordinate are included and four-dimensional matter may be viewed as a manifestation of empty five-dimensional geometry. In effect, the geometry induces the matter. The extension of this interpretation to arbitrary dimensions has been considered in Ref. [28]. It would be interesting to consider the relationship

between the RR fields and eleven-dimensional geometry further in this context, although this is beyond the scope of the present paper.

To conclude, we consider the class of NS–NS backgrounds:

$$^{(10)}ds_{\text{SSW}}^2 = 2dudv + A_u(x^i)du^2 + A_i(x^i)dx^i du - \delta_{ij}dx^i dx^j, \quad (33)$$

where the dilaton is fixed ($\Phi = 0$), the vector function A_μ is independent of u and v and the ‘chiral’ constraint $A_i = -2B_i^{(1)}$ is imposed. This latter constraint implies that the vector-dependent terms in the graviton β -function (19) cancel and this equation reduces to $\Delta A_u = 0$. The background (33) may be embedded in the type IIA and type IIB superstring theories [21]. In the zero-slope limit, it corresponds to a supersymmetric string wave that admits eight constant Killing spinors and has precisely one-half of the spacetime supersymmetries unbroken [15]. Thus, the embeddings that we have considered in this paper apply to this class of supersymmetric string waves. Indeed, since $M^2 = 4N^2$, it follows from Eq. (13) that $^{(10)}R_{uu} = N^2/4 = \text{constant}$ and the embedding metric is therefore given by

$$^{(11)}ds_{\text{embed}}^2 = ^{(10)}ds_{\text{SSW}}^2 - \varphi^2 d\psi^2, \quad (34)$$

where φ is determined by Eq. (22).

The dimensional reduction of $N = 1$, $n = 11$ supergravity on a circle results in type IIA supergravity [21]. This feature has recently been employed to generate a new solution to the field equations of eleven-dimensional supergravity by lifting the ten-dimensional type IIA solution (33) to eleven dimensions [21]. The new solution generated in this fashion is

$$^{(11)}ds_{\text{M}}^2 = ^{(10)}ds_{\text{SSW}}^2 - dy^2, \quad (35)$$

where y represents the coordinate of the eleventh dimension. The three-form antisymmetric potential is determined by the components $A_i(x^i)$. Eq. (35) generalizes the pp-wave solution found by Hull [29]. Since it admits a covariantly constant null Killing vector field, we may consider its embedding in a twelve-dimensional, Ricci-flat space along the lines discussed above. In particular, the only non-trivial component of the Ricci-tensor $^{(11)}R^\alpha_\beta$ is $^{(11)}R^v_u = ^{(10)}R_{uu} = \text{constant}$, where $^{(10)}R_{uu}$ is calculated from the metric (33). This implies that we may immediately write down the embedding of the metric (35) in a twelve-dimensional, Ricci-flat space:

$$^{(12)}ds^2 = ^{(10)}ds_{\text{SSW}}^2 - dy^2 - \varphi^2 d\psi^2, \quad (36)$$

where φ is again given by Eq. (22).

An important consequence of Campbell's theorem is that once the embedding of a n -dimensional manifold in a $(n + 1)$ -dimensional, Ricci-flat space has been established, the procedure may be repeated indefinitely to embed the spacetime in Ricci-flat spaces of dimension $d \geq n + 2$. Hence, the type IIA supersymmetric string wave (33) may be embedded in twelve-dimensional Einstein gravity by embedding the eleven-dimensional manifold (34). Since this space is itself Ricci-flat, Eqs. (3)–(8) are solved by $\Omega_{\alpha\beta} = 0$ and $\varphi = 1$. This implies that the twelve-dimensional embedding metric is given by $^{(12)}ds^2 = ^{(11)}ds_{\text{embed}}^2 - dw^2$, where w represents the coordinate of the twelfth dimension. However, this is formally identical to the metric (36) if the extra dimensions are identified in an appropriate fashion. We remark that the twelfth dimension may be either space-like or time-like in both cases. Thus, the ten-dimensional type IIA supersymmetric string wave background and the solution (35) to eleven-dimensional supergravity may be locally embedded in the same twelve-dimensional, Ricci-flat space.

In conclusion, therefore, we have employed Campbell's theorem to establish the local embedding of a general class of exact, ten-dimensional superstring backgrounds in eleven-dimensional, Ricci-flat spaces. The embedded backgrounds admit a covariantly constant null Killing vector field and the embedding spaces represent exact solutions of eleven-dimensional, vacuum Einstein gravity. This is interesting because eleven-dimensional general relativity may be directly related at a certain level to ten-dimensional superstring theory. There has been widespread interest recently in the possibility that the five separate superstring theories are related by discrete duality symmetries at a non-perturbative level [21, 22, 23, 30, 31, 32]. This has motivated the conjecture that they arise from a more fundamental quantum theory (M-theory) [32, 33]. Although the precise form of such a theory is at present unknown, its low-energy effective action is eleven-dimensional supergravity with a vacuum limit given by the Einstein–Hilbert action [34]. In this sense, therefore, Campbell's theorem provides a potential link between vacuum solutions of M-theory and ten-dimensional superstring backgrounds that are valid in the strong curvature and strong coupling regimes.

It would be of interest to establish similar embeddings for other classes of exact string solutions. In particular, we have considered backgrounds where the dilaton is a function of

u only, but it may also depend on the transverse coordinates x^i . When the metric admits a covariantly constant null Killing vector field and all massless excitations in the NS-NS sector are non-trivial functions of x^i only, the backgrounds (1) correspond to the class of ‘ K -models’ considered in Refs. [14, 16]. There is also the related class of ‘ F -models’ that are characterized by two null Killing vectors and a chiral coupling between $^{(10)}g_{uv}$ and $B_{uv}^{(1)}$ [13, 14]. The simplest F -model is determined by a single function $F(x^i)$, where

$$^{(10)}ds^2 = F(x^i)dudv - \delta_{ij}dx^i dx^j, \quad \Delta F^{-1} = 0 \quad (37)$$

and included in this class is the fundamental string solution [35]. Although Campbell’s theorem implies that the embedding of these F -models is possible in principle, the form of the embedding will be more complicated than that considered in this work because the Ricci curvature scalar of these backgrounds is in general non-zero.

Finally, it is worth remarking that Campbell’s theorem implies that the type IIB backgrounds can also be embedded in twelve-dimensional, Ricci-flat spaces. It would be interesting to explore possible consequences of this feature within the context of the recently proposed ‘ F -theory’ [31]. It has been conjectured that the dimensional reduction of this twelve-dimensional theory on a two-torus reproduces the type IIB theory, thereby providing a geometrical interpretation of the $SL(2, Z)$ symmetry of the theory in ten dimensions. The interpretation of the D-instanton background of the type IIB theory as the dimensional reduction of a twelve-dimensional gravitational wave has recently been discussed by Tseytlin [36]. The question arises as to whether other ten-dimensional backgrounds may be interpreted in a similar fashion by embedding them in twelve dimensions. This may provide further insight into the origin of the RR fields χ and $B_{\mu\nu}^{(2)}$.

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